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LETTER TO THE EDITOR

Reflection quadratic algebra associated with Z_2 model

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Abstract. The reflection quadratic algebra $\mathscr{A}(R)$ associated with the Z_2 model is formulated from the reflection equation. It is proved that $\mathscr{A}(R)$ is isomorphic to the matrix-element algebra $A(R)$ of the quantum group related to R, and thus its coalgebra structure is obtained. A new $A(R)$-comodule structure of $\mathscr{A}(R)$ is revealed.

In addition to the Yang–Baxter equations, quantum groups and quantum algebras arising from quantum integrability through the Faddeev–Reshetikhin–Takhtajin approach, the list of algebraic objects was enriched with a new item: the reflection equation and its related reflection quadratic algebra, which are introduced in [1] as an equation describing factoring scattering on a halfline. Recently they were also applied to the quantum current algebras [2] and to the integrable modules with non-periodic boundary conditions [3, 4]. Kulish *et al* [5] studied the properties of some quadratic algebras (including some representations) and constructed the constant solutions of the reflection equations [6].

The reflection equation (without spectral parameter) we shall consider in this letter reads

$$RK^1R^{t_1}K^2=K^2R^{t_1}K^1R \tag{1}$$

where K is a square matrix, $K^1=K\otimes \mathrm{id}$, $K^2=\mathrm{id}\otimes K$, and the superscript t_1 denotes transposition in the first space. The reflection quadratic algebra is an associative algebra generated by the non-commuting matrix elements of the matrix K and unit 1.

In this letter we are devoted to the reflection quadratic algebra related to the Z_2 (eight-vertex) models. We shall see that this algebra has an interesting property: it is isomorphic to the matrix-element algebra of quantum group related to Z_2 model. Using this property we endow the reflection quadratic algebra with the coalgebra structure, and then its new quantum-group comodule structure.

In this letter we denote by C the complex number field.

The R-matrix of the Z_2 model we consider in this letter is

$$R=\begin{bmatrix}1&&&t\\&\omega t&1&\\&1&\omega t&\\t&&&1\end{bmatrix} \tag{2}$$

where $\omega^2=1$. Suppose that K is of the form

$$K=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \tag{3}$$

and R, K satisfy the reflection equation [1]. The reflection quadratic algebra $\mathscr{A}(R)$ is defined as an associated algebra over C generated by α, β, γ, δ and the unit 1 subject to the relations [1].

Proposition 1. If $t^2\neq 1$, then the algebra $\mathscr{A}(R)$ is an associative algebra generated by α, β, γ, δ, 1, with the following relations:

$$\alpha^2=\delta^2 \qquad [\beta,\gamma]=0 \tag{4}$$

$$\beta^2=\gamma^2 \qquad [\alpha,\delta]=0 \tag{5}$$

$$\delta\beta=\omega\alpha\gamma \qquad \beta\alpha=\omega\gamma\delta \tag{6}$$

$$\beta\delta=\omega\gamma\alpha \qquad \alpha\beta=\omega\delta\gamma. \tag{7}$$

Proof of this proposition is straightforward. In fact, expanding the matrix equation (1) and comparing its matrix elements, we obtain the following equations

$$\omega t(\gamma^2-\beta^2)+[\delta,\alpha]=0 \tag{8}$$

$$(\gamma^2-\beta^2)+\omega t[\delta,\alpha]=0 \tag{9}$$

$$t(\alpha^2-\delta^2)+[\beta,\gamma]=0 \tag{10}$$

$$(\alpha^2-\delta^2)+t[\beta,\gamma]=0 \tag{11}$$

$$-t^2\beta\alpha+t\delta\beta-\omega t\alpha\gamma+\omega t^2\gamma\delta=0 \tag{12}$$

$$t\beta\alpha-t^2\delta\beta+\omega t^2\alpha\gamma-\omega t\gamma\delta=0 \tag{13}$$

$$t^2\delta\gamma-\omega t\beta\delta+t\gamma\alpha-\omega t^2\alpha\beta=0 \tag{14}$$

$$-\omega t\delta\gamma+t^2\beta\delta-\omega t^2\gamma\alpha+t\alpha\beta=0. \tag{15}$$

From (8, 9), (10, 11), (12, 13), and (14, 15), we obtain the defining relations (4), (5), (6), and (7), respectively.

The constant solution of the reflection equation (1) can be easily derived by regarding α, β, γ, δ as the complex numbers. As results these constant solutions read

$$K^{(1)}=\begin{bmatrix}\alpha & \beta\\ \omega\beta & \alpha\end{bmatrix} \qquad K^{(2)}=\begin{bmatrix}\alpha & \beta\\ -\omega\beta & -\alpha\end{bmatrix} \tag{16}$$

where α and β are arbitrary complex numbers.

It is well known that $\mathscr{A}(R)$ is an $A(R)$-comodule, i.e. there exists an algebraic homomorphism $\varphi:\mathscr{A}(R)\rightarrow A(R)\otimes\mathscr{A}(R)$ such that

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\varphi&=(\mathrm{id}\otimes\varphi)\circ\varphi\\ (\varepsilon\otimes\mathrm{id})\circ\varphi&=\mathrm{id}\end{aligned} \tag{17}$$

where Δ and ε are the coproduct and the co-unit of $A(R)$, respectively. In fact, φ is explicitly defined by [5, 6]

$$\begin{aligned}\varphi(K)&=TKT^t\\ \varphi(K)_{ij}&=\sum_{m,n} t_{im}t_{jn}k_{mn}\end{aligned} \tag{18}$$

provided $[t_{ij}, k_{mn}] = 0$. This property implies that, if K is a solution of equation (1), then $\varphi(K)$ is also a solution. We note that $\mathscr{A}(R)$ is a *two-side* $A(R)$-comodule. Such a structure is standard. However, for the case in hand, we can endow $\mathscr{A}(R)$ with a new $A(R)$-comodule structure.

We first study the explicit form of $A(R)$, which is an associative algebra generated by a, b, c, d and 1 subject to

$$RT^1T^2 = T^2T^1R \tag{19}$$

where $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $T^1 = T \otimes \mathrm{id}$, $T^2 = \mathrm{id} \otimes T$. Comparing two sides of the equation (19), we immediately obtain proposition 2.

Proposition 2. If $t^2 \neq 1$, associative algebra $A(R)$ is generated by a, b, c, d and 1 satisfying the relations

$$\begin{array}{llll} a^2 = d^2 & b^2 = c^2 & ad = da & bc = cb \\ ab = \omega dc & ba = \omega cd & & \\ ac = \omega db & ca = \omega bd. & & \end{array} \tag{20}$$

From Proposition 1 and 2, the following result is obvious.

Proposition 3. If $t^2 \neq 1$, the mapping ψ: $\mathscr{A}(R) \to A(R)$ defined by

$$\alpha \mapsto a, \beta \mapsto b, \gamma \mapsto c, \delta \mapsto d \tag{21}$$

is an algebraic isomorphism.

One can easily check that the quantum group $A(R)$ has standard coproduct Δ and co-unit ε

$$\Delta(T) = T \otimes T \qquad \varepsilon(T) = I \tag{22}$$

or explicitly,

$$\begin{array}{ll} \Delta(a) = a \otimes a + b \otimes c & \Delta(b) = a \otimes b + b \otimes d \\ \Delta(c) = c \otimes a + d \otimes c & \Delta(d) = c \otimes b + d \otimes d \\ \varepsilon(a) = \varepsilon(d) = 0 & \\ \varepsilon(b) = \varepsilon(c) = \varepsilon(1) = 1. & \end{array} \tag{23}$$

According to proposition 3 we can also endow reflection algebra $\mathscr{A}(R)$ with coproduct $\Delta_{\mathscr{A}}$ and co-unit $\varepsilon_{\mathscr{A}}$

$$\begin{aligned} \Delta_{\mathscr{A}} &= (\psi^{-1} \otimes \psi^{-1})\Delta_A \psi \\ \varepsilon_{\mathscr{A}} &= \varepsilon_A \psi \end{aligned} \tag{24}$$

where $\Delta_{\mathscr{A}}$ and ε_A are the coproduct and co-unit of $A(R)$, respectively, given in equations (22) and (23).

Recall that, for an algebra, one can define itself as its regular (left) module through its product. Correspondingly, for a coalgebra, one can define itself as its *regular* (left) comodule through the coproduct. For the case in hand, we can identify the algebras $A(R)$ and $\mathscr{A}(R)$ from proposition 3. Then we can define a new $A(R)$-comodule structure of $\mathscr{A}(R)$, which is virtually the *regular* comodule structure of $\mathscr{A}(R)$.

Define

$$\varphi = (\psi \otimes \mathrm{id})\Delta_{\mathscr{A}} : \mathscr{A}(R) \to A(R) \otimes \mathscr{A}(R). \tag{25}$$

Then one can prove that φ defined in (25) indeed satisfy the relation (17). In fact

$$\begin{aligned}(\Delta_A \otimes \mathrm{id})\varphi &= (\Delta_A \otimes \mathrm{id})(\psi \otimes \mathrm{id})\Delta_{\mathscr{A}} \\ &= (\psi \otimes \psi \otimes \mathrm{id})(\Delta_{\mathscr{A}} \otimes \mathrm{id})\Delta_{\mathscr{A}} \\ &= (\psi \otimes \psi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{\mathscr{A}})\Delta_{\mathscr{A}}[\psi \otimes (\psi \otimes \mathrm{id})\Delta_{\mathscr{A}}]\Delta_{\mathscr{A}} \\ &= (\mathrm{id} \otimes (\psi \otimes \mathrm{id})\Delta_{\mathscr{A}})(\psi \otimes \mathrm{id})\Delta_{\mathscr{A}} = (\mathrm{id} \otimes \varphi)\varphi \\ (\varepsilon_A \otimes \mathrm{id})\varphi &= (\varepsilon_A \otimes \mathrm{id})(\phi \otimes \mathrm{id})\Delta_{\mathscr{A}} = (\varepsilon_A \phi \otimes \mathrm{id})\Delta_{\mathscr{A}} \\ &= (\varepsilon_{\mathscr{A}} \otimes \mathrm{id})\Delta_{\mathscr{A}} = \mathrm{id}\end{aligned} \tag{26}$$

where we have used the relation

$$\Delta_A \psi = (\psi \otimes \psi)\Delta_{\mathscr{A}}. \tag{27}$$

Therefore φ defines a *new* $A(R)$-comodule structure of $\mathscr{A}(R)$, in which elements of $A(R)$ do not commute with those of $\mathscr{A}(R)$. We would like to point out that this comodule is only a *left* comodule, not the *two-side* comodule.

So far we have studied the reflection quadratic algebra related to Z_2 module and its new quantum-group-comodule structure. In [7] the R-matrix for Z_n model was formulated. Using this we can further study the reflection quadratic algebras related to Z_n model. An interesting question is, whether the reflection quadratic algebra is isomorphic to the matrix-element algebra of corresponding quantum group or not. We shall consider this question in a separate paper.

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